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Semi-linear Liouville theorems in the Heisenberg group via vector field methods ☆

Lu Xu

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, PR China

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ABSTRACT

We consider a global Liouville type theorem for semi-linear elliptic equation on Heisenberg group. Our main idea is the method of integration by parts for a "vector field" analogous to the quantity in Gidas and Spruck (1981) [5].

1. Introduction and main result

An important feature of this paper is that we develop the analogue of the vector field method for the Heisenberg group. This allows us to obtain some new non-existence results for a class of positive solutions of the following semi-linear equation

$$\Delta_H u + h(x)u^p = 0 \quad \text{in } H^n, \tag{1.1}$$

where Δ_H is the Heisenberg Laplacian, H^n is the Heisenberg group whose underlying manifold is $\mathbb{R}^{2n+1}$ $(n \geqslant 1)$, and $h \geqslant 0$, $1 < p < 1 + (8n+7)/(2n+1)^2$. We consider H^n as the set $\mathbb{C}^n \times \mathbb{R}$ with coordinates (z,t) and the group law

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E-mail address: xulu@wipm.ac.cn.

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$$(z,t)(\xi,\tau)=\left(z+\xi,t+\tau+2\operatorname{Im}\sum_{\alpha=1}^{n}z^\alpha\bar{\xi}^\alpha\right)\quad\text{for } (z,t),(\xi,\tau)\in\mathbb{C}^n\times\mathbb{R}, \tag{1.2}$$

and the norm

$$\left|(z,t)\right|=\left(t^2+|z|^4\right)^{\frac{1}{4}}. \tag{1.3}$$

The operator Δ_H is the subelliptic Laplacian on H^n defined by

$$\Delta_H=\sum_{\alpha=1}^{n}(Z_\alpha Z_{\bar{\alpha}}+Z_{\bar{\alpha}}Z_\alpha) \tag{1.4}$$

with the left-invariant vector fields $Z_\alpha=\partial/\partial z^\alpha+i\bar{z}^\alpha\partial/\partial t$, $\alpha=1,\dots,n$.

Denoting by $Q=2n+2$ the homogeneous dimension of H^n, our main result is the following global Liouville type theorem.

Theorem 1. *Let $u(x)\in C^2(H^n)$ be a nonnegative solution of*

$$\Delta_H u+h(x)u^p=0\quad \text{in } H^n,\ n>1, \tag{1.5}$$

with $1<p<\frac{Q(Q+2)}{(Q-1)^2}$. Assume that $h(x)$ is a nonnegative function such that

$$\Delta_H h(x)\geqslant 0\quad \text{in } H^n, \tag{1.6}$$

and for $|x|$ large,

$$\left|\nabla\log h(x)\right|\leqslant\frac{c}{|x|}, \tag{1.7}$$

$$c_1|x|^\sigma\leqslant h(x),\quad \sigma>-\frac{5n+1}{4n^2+2n-3}. \tag{1.8}$$

Then $u(x)\equiv 0$.

Theorem 1 extends the Liouville theorems of Birindelli, Dolcetta and Cutri [1]. Indeed one of the results of [1] was that for $1<p\leqslant\frac{Q}{Q-2}$, the only nonnegative solutions of

$$\Delta_H u+u^p=0\quad \text{in } H^n \tag{1.9}$$

are the trivial ones. There, as well as here, no conditions at infinity for u are required. To our knowledge, non-existence results for Eq. (1.9) were stated as an open problem for $\frac{Q}{Q-2}<p<\frac{Q+2}{Q-2}$, except for special cases, such as the solutions are cylindrical or decay at infinity, see the work [2] and references therein. When $p=\frac{Q+2}{Q-2}$, the uniqueness theorem for positive entire solutions of (1.9) has been studied by Jerison and Lee in [7,8] under $u\in L^{1+p}(H^n)$ and later by Garofalo and Vassilev in [6] for cylindrical symmetric solutions.

Remark 1. Let us point out

$$\frac{Q}{Q-2}<\frac{Q(Q+2)}{(Q-1)^2}<\frac{Q+2}{Q-2}, \tag{1.10}$$

which implies our result gives a partial affirmative answer to the above open problem when $n > 1$. For $n = 1$, it may be an interesting open problem.

In the Euclidean case, the non-existence results for $1 \leqslant p < \frac{n+2}{n-2}$, have first been proved by Gidas and Spruck in [5] and then by Chen and Li in [3] using the method of moving plane. Gidas and Spruck [5] introduced the vector field for semi-linear elliptic equation in $\mathbb{R}^n$ and some manifolds. The basic form of vector field appeared in a geometric result of Obata [9] concerning deformations of the usual metric on $\mathbb{S}^n$. In fact, the vector field leads to some identities which were essential for their work. Using this idea, S.Y.A. Chang, M. Gursky and P. Yang in [4] considered the classification of entire solutions of a fully nonlinear equation in conformal geometry.

In the second section, we introduce a quantity which comes from vector field similar to Obata or Gidas and Spruck [5], then give out an integral about this quantity in two expressions. Let us point out that the difference in the computation with respect to the Euclidean case is that the Hessian is not symmetric and hence there are terms that appear from the commutator. In the last section we prove our main theorem by dealing with the integral delicately.

2. An integral about a vector field of Obata

In the rest of this paper, we will always use the standard holomorphic frame $\{Z_i\}$ and dual admissible co-frame $\{dz^i\}$ for $i = 1, \dots, n$. Covariant derivatives on H^n are given by $u_{j_1 \dots j_r} = Z_{j_r} \cdots Z_{j_1} u$, for $j_s = i, \bar{i}$, or 0, with the conventions that $Z_0 = \partial/\partial t$ and $Z_{\bar{i}} = \overline{Z}_i$. The second covariant derivatives of a scalar function u satisfy the following commutation relations:

$$u_{ij} - u_{ji} = 0, \tag{2.1}$$

$$u_{i\bar{j}} - u_{\bar{j}i} = 2i\delta_{i\bar{j}} u_0, \tag{2.2}$$

$$u_{0i} - u_{i0} = 0. \tag{2.3}$$

Next let $u \geqslant 0$ satisfy (1.5). Combining with (2.2) we get

$$\sum_{i=1}^{n} u_{i\bar{i}} = niu_0 - \frac{h}{2} u^p, \tag{2.4}$$

$$\sum_{i=1}^{n} u_{\bar{i}i} = -niu_0 - \frac{h}{2} u^p. \tag{2.5}$$

Set $u = v^{-k}$ $(k \neq 0)$, then

$$u_i = -kv^{-k-1} v_i, \tag{2.6}$$

$$u_{i\bar{j}} = -kv^{-k-1} v_{i\bar{j}} + k(k+1) v^{-k-2} v_i v_{\bar{j}}, \tag{2.7}$$

$$\Delta_H u = -kv^{-k-1} \Delta_H v + 2k(k+1) v^{-k-2} |\nabla v|^2, \tag{2.8}$$

where $|\nabla v|^2 = \sum_{i=1}^{n} v_i v_{\bar{i}}$. Thus from (1.5), v satisfies the equation

$$\Delta_H v = 2(k+1) v^{-1} |\nabla v|^2 + \frac{h}{k} v^{k+1-kp}. \tag{2.9}$$

Again by (2.2) for v, $\sum_{i=1}^{n} (v_{i\bar{i}} - v_{\bar{i}i}) = 2niv_0$, then

$$\begin{cases} \sum_{i=1}^{n} v_{i\bar{i}} = niv_0 + (k+1)v^{-1}|\nabla v|^2 + \dfrac{h}{2k}v^{k+1-kp}, \\ \sum_{i=1}^{n} v_{\bar{i}i} = -niv_0 + (k+1)v^{-1}|\nabla v|^2 + \dfrac{h}{2k}v^{k+1-kp}. \end{cases} \tag{2.10}$$

And by (2.4) and (2.5), we also get

$$\begin{cases} \sum_{i=1}^{n} u_{i\bar{i}} = ni\left(v^{-k}\right)_0 - \dfrac{h}{2}v^{-pk} = -kniv^{-k-1}v_0 - \dfrac{h}{2}v^{-pk}, \\ \sum_{i=1}^{n} u_{\bar{i}i} = kniv^{-k-1}v_0 - \dfrac{h}{2}v^{-pk}. \end{cases} \tag{2.11}$$

We define a similar "vector field" as in [5,9]:

$$E^u_{i\bar{j}} =: u_{i\bar{j}} - \frac{1}{n}\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}}\delta_{i\bar{j}}. \tag{2.12}$$

Obviously $E^u_{i\bar{j}}$ is trace free, i.e.

$$\sum_{i=1}^{n} E^u_{i\bar{i}} = \sum_{i,j=1}^{n} \delta_{i\bar{j}}E^u_{i\bar{j}} = \sum_{i=1}^{n} u_{i\bar{i}} - \sum_{i=1}^{n} u_{i\bar{i}} \sum_{i,j=1}^{n} \frac{\delta_{i\bar{j}}}{n} = 0. \tag{2.13}$$

From (2.7) and (2.12),

$$E^u_{i\bar{j}} = k(k+1)v^{-k-2}\left(v_i v_{\bar{j}} - \frac{1}{n}|\nabla v|^2\delta_{i\bar{j}}\right) - kv^{-k-1}E^v_{i\bar{j}}, \tag{2.14}$$

and $E^v_{i\bar{j}}$ is also trace free.

Given a bounded domain $\Omega \subset \mathbb{R}^{2n+1}$ $(n > 1)$, let $\varphi \in C_0^\infty(\Omega)$, $0 \leqslant \varphi \leqslant 1$. Now we consider the nonnegative integral

$$\sum_{i,j=1}^{n} \int_\Omega \left|E^u_{i\bar{j}}\right|^2 \varphi^q v^{r+2k+2}, \tag{2.15}$$

where q, k, r are to be determined. We will express the integral in two forms by the following two steps respectively. For convenience, we omit the domain Ω in all integrals below.

Step I. On the one hand, by (2.14) and $E^v_{i\bar{j}}$ is trace free,

$$\begin{aligned} &\sum_{i,j=1}^{n} \int \left|E^u_{i\bar{j}}\right|^2 \varphi^q v^{r+2k+2} \\ &\quad= \sum_{i,j=1}^{n} \int E^u_{i\bar{j}} \cdot \overline{E^u_{i\bar{j}}}\varphi^q v^{r+2k+2} \\ &\quad= \sum_{i,j=1}^{n} \int \left[k(k+1)v^{-k-2}\left(v_i v_{\bar{j}} - \frac{1}{n}|\nabla v|^2\delta_{i\bar{j}}\right) - kv^{-k-1}E^v_{i\bar{j}}\right] \end{aligned}$$

$$
\begin{aligned}
&\times\left[k(k+1)v^{-k-2}\left(v_{\bar{i}}v_j-\frac{1}{n}|\nabla v|^2\delta_{\bar{i}j}\right)-kv^{-k-1}E^v_{\bar{i}j}\right]\varphi^q v^{r+2k+2}\\
&=\sum_{i,j=1}^n k^2\int\left|E^v_{i\bar{j}}\right|^2\varphi^q v^r+\left(1-\frac{1}{n}\right)k^2(k+1)^2\int v^{r-2}\varphi^q|\nabla v|^4\\
&\quad-k^2(k+1)\sum_{i,j=1}^n\int v^{r-1}\varphi^q\left(v_iv_{\bar{j}}E^v_{\bar{i}j}+v_{\bar{i}}v_jE^v_{i\bar{j}}\right)\\
&=: I+II+III. \qquad (2.16)
\end{aligned}
$$

Preserving I and II, we calculate

$$
\begin{aligned}
III&=-k^2(k+1)\sum_{i,j=1}^n\int v^{r-1}\varphi^q\left[v_iv_{\bar{j}}\left(v_{\bar{i}j}-\frac{1}{n}\sum_{\gamma=1}^n v_{\bar{\gamma}\gamma}\delta_{\bar{i}j}\right)\right.\\
&\quad\left.+v_{\bar{i}}v_j\left(v_{i\bar{j}}-\frac{1}{n}\sum_{\gamma=1}^n v_{\gamma\bar{\gamma}}\delta_{i\bar{j}}\right)\right]\\
&=\frac{k^2(k+1)}{n}\int v^{r-1}\varphi^q|\nabla v|^2\Delta_H v\\
&\quad-k^2(k+1)\sum_{i,j=1}^n\int v^{r-1}\varphi^q(v_iv_{\bar{j}}v_{\bar{i}j}+v_{\bar{i}}v_jv_{i\bar{j}})\\
&=: III_1+III_2. \qquad (2.17)
\end{aligned}
$$

And further we have

$$
\begin{aligned}
III_2&=-k^2(k+1)\sum_{i,j=1}^n\int v^{r-1}\varphi^q(v_iv_{\bar{j}}v_{\bar{i}j}+v_{\bar{i}}v_jv_{i\bar{j}})\\
&=-k^2(k+1)\sum_{j=1}^n\int v^{r-1}\varphi^q\left\{\left[\left(|\nabla v|^2\right)_j-\sum_{i=1}^n v_{\bar{i}}v_{ij}\right]v_{\bar{j}}\right.\\
&\quad\left.+\left[\left(|\nabla v|^2\right)_{\bar{j}}-\sum_{i=1}^n v_iv_{\bar{i}\bar{j}}\right]v_j\right\}.
\end{aligned}
$$

Using integration by parts, it gives

$$
\begin{aligned}
III_2&=k^2(k+1)\sum_{j=1}^n\int\varphi^q|\nabla v|^2\left[\left(v^{r-1}\right)_jv_{\bar{j}}+\left(v^{r-1}\right)_{\bar{j}}v_j\right]\\
&\quad+k^2(k+1)\sum_{j=1}^n\int v^{r-1}|\nabla v|^2\left[\left(\varphi^q\right)_jv_{\bar{j}}+\left(\varphi^q\right)_{\bar{j}}v_j\right]\\
&\quad+k^2(k+1)\sum_{j=1}^n\int v^{r-1}|\nabla v|^2\varphi^q(v_{\bar{j}j}+v_{j\bar{j}})
\end{aligned}
$$

$$
\begin{aligned}
&+k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q(v_{\bar i}v_{\bar j}v_{ij}+v_iv_jv_{\bar i\bar j})\\
&=k^2(k+1)\int v^{r-1}\varphi^q|\nabla v|^2\Delta_H v+2k^2(k+1)(r-1)\int v^{r-2}\varphi^q|\nabla v|^4\\
&\quad+k^2(k+1)\sum_{j=1}^{n}\int v^{r-1}|\nabla v|^2\big[\big(\varphi^q\big)_j v_{\bar j}+\big(\varphi^q\big)_{\bar j}v_j\big]\\
&\quad+k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q(v_{\bar i}v_{\bar j}v_{ij}+v_iv_jv_{\bar i\bar j}).
\end{aligned}\tag{2.18}
$$

Substituting Eq. (2.9) for $\Delta_H v$, we finally obtain from (2.16) to (2.18) that

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\int\big|E^u_{i\bar j}\big|^2\varphi^q v^{r+2k+2}\\
&\quad=k^2\sum_{i,j=1}^{n}\int\big|E^v_{i\bar j}\big|^2\varphi^q v^r+\left(1+\frac{1}{n}\right)k(k+1)\int h v^{r+k-pk}\varphi^q|\nabla v|^2\\
&\qquad+k^2(k+1)\left[(k+1)\left(3+\frac{1}{n}\right)+2(r-1)\right]\int v^{r-2}\varphi^q|\nabla v|^4\\
&\qquad+k^2(k+1)\sum_{j=1}^{n}\int v^{r-1}|\nabla v|^2\big[\big(\varphi^q\big)_j v_{\bar j}+\big(\varphi^q\big)_{\bar j}v_j\big]\\
&\qquad+k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q(v_{\bar i}v_{\bar j}v_{ij}+v_iv_jv_{\bar i\bar j}).
\end{aligned}\tag{2.19}
$$

Step II. On the other hand, using (2.12) and the fact $E^u_{i\bar j}$ is trace free, we easily get

$$
\begin{aligned}
\sum_{i,j=1}^{n}\int\big|E^u_{i\bar j}\big|^2\varphi^q v^{r+2k+2}&=\sum_{i,j=1}^{n}\int E^u_{i\bar j}\left[u_{\bar i j}-\frac{1}{n}\sum_{\gamma=1}^{n}u_{\bar\gamma\gamma}\delta_{\bar i j}\right]\varphi^q v^{r+2k+2}\\
&=\sum_{i,j=1}^{n}\int E^u_{i\bar j}u_{\bar i j}\varphi^q v^{r+2k+2},
\end{aligned}
$$

which gives after integration by parts

$$
\begin{aligned}
\sum_{i,j=1}^{n}\int\big|E^u_{i\bar j}\big|^2\varphi^q v^{r+2k+2}&=-\sum_{i,j=1}^{n}\int\big(E^u_{i\bar j}\big)_j u_{\bar i}\varphi^q v^{r+2k+2}-\sum_{i,j=1}^{n}\int E^u_{i\bar j}u_{\bar i}\big(\varphi^q\big)_j v^{r+2k+2}\\
&\quad-\sum_{i,j=1}^{n}\int E^u_{i\bar j}u_{\bar i}\varphi^q\big(v^{r+2k+2}\big)_j\\
&=: IV+V+VI.
\end{aligned}\tag{2.20}
$$

Next we deal with IV–VI. Firstly

$$\begin{aligned}
IV &= -\sum_{i,j=1}^{n}\int \left(E^{u}_{i\bar{j}}\right)_{j} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= -\sum_{i,j=1}^{n}\int \left(u_{i\bar{j}} - \frac{1}{n}\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}}\delta_{i\bar{j}}\right)_{j} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= -\sum_{i,j=1}^{n}\int u_{i\bar{j}j} u_{\bar{i}}\varphi^{q} v^{r+2k+2} + \frac{1}{n}\sum_{i=1}^{n}\int \left(\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}}\right)_{i} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= -\sum_{i,j=1}^{n}\int [u_{\bar{j}ij} + 2i\delta_{i\bar{j}}u_{0i}] u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\int \left(\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}}\right)_{i} u_{\bar{i}}\varphi^{q} v^{r+2k+2},
\end{aligned} \tag{2.21}$$

here and below we use relations (2.1)–(2.3).

$$\begin{aligned}
IV &= -\sum_{i,j=1}^{n}\int [u_{j\bar{j}i} - 2i\delta_{j\bar{j}}u_{0i} + 2i\delta_{i\bar{j}}u_{0j}] u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\int \left(\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}}\right)_{i} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= \left(\frac{1}{n}-1\right)\sum_{i=1}^{n}\int \left(\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}}\right)_{i} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&\quad + 2(n-1)i\sum_{i=1}^{n}\int u_{0i}u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= \left(\frac{1}{n}-1\right)\sum_{i=1}^{n}\int \left(niu_{0} - \frac{h}{2}u^{p}\right)_{i} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&\quad + 2(n-1)i\sum_{i=1}^{n}\int u_{0i}u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= (n-1)i\sum_{i=1}^{n}\int u_{0i}u_{\bar{i}}\varphi^{q} v^{r+2k+2} + \frac{n-1}{2n}\sum_{i=1}^{n}\int \left(hu^{p}\right)_{i} u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&=: i(n-1)IV_{1} + \frac{n-1}{2n}IV_{2}.
\end{aligned} \tag{2.22}$$

Using integration by parts for i and (2.11),

$$\begin{aligned}
IV_{1} &= \sum_{i=1}^{n}\int u_{0i}u_{\bar{i}}\varphi^{q} v^{r+2k+2} \\
&= -\sum_{i=1}^{n}\left[\int u_{0}u_{\bar{i}i}\varphi^{q} v^{r+2k+2} - \int u_{0}u_{\bar{i}}\left(\varphi^{q}\right)_{i} v^{r+2k+2} - \int u_{0}u_{\bar{i}}\varphi^{q}\left(v^{r+2k+2}\right)_{i}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\int \left(-kv^{-k-1}v_0\right)\left(nkiv^{-k-1}v_0 - \frac{h}{2}v^{-pk}\right)\varphi^q v^{r+2k+2} \\
&\quad - \sum_{i=1}^{n} \int \left(-kv^{-k-1}v_0\right)\left(-kv^{-k-1}v_{\bar{i}}\right)\left(\varphi^q\right)_i v^{r+2k+2} \\
&\quad - (r+2k+2)\sum_{i=1}^{n} \int \left(-kv^{-k-1}v_0\right)\left(-kv^{-k-1}v_{\bar{i}}\right)\varphi^q v^{r+2k+1} v_i \\
&= nk^2 i\int v^r\varphi^q v_0^2 - \frac{k}{2}\int hv^{r+k+1-pk}\varphi^q v_0 \\
&\quad - k^2\sum_{i=1}^{n}\int v^r\left(\varphi^q\right)_i v_{\bar{i}} v_0 - (r+2k+2)k^2\int v^{r-1}\varphi^q|\nabla v|^2 v_0, \qquad (2.23)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
IV_2 &= \sum_{i=1}^{n}\int \left(hu^p\right)_i u_{\bar{i}}\varphi^q v^{r+2k+2} \\
&= -nki\int hv^{r+k-1-pk}\varphi^q v_0 + \frac{1}{2}\int h^2 v^{r+2k+2-2pk}\varphi^q \\
&\quad + k\sum_{i=1}^{n}\int hv^{r+k+1-pk}v_{\bar{i}}\left(\varphi^q\right)_i + k(r+2k+2)\int hv^{r+k-pk}\varphi^q|\nabla v|^2. \qquad (2.24)
\end{aligned}
$$

Thus

$$
\begin{aligned}
IV &= -n(n-1)k^2\int v^r\varphi^q v_0^2 - k(n-1)i\int hv^{r+k+1-pk}\varphi^q v_0 \\
&\quad + \frac{k(r+2k+2)(n-1)}{2n}\int hv^{r+k-pk}\varphi^q|\nabla v|^2 \\
&\quad + \frac{n-1}{4n}\int h^2 v^{r+2k+2-2pk}\varphi^q \\
&\quad + \frac{k(n-1)}{2n}\sum_{i=1}^{n}\int hv^{r+k+1-pk}v_{\bar{i}}\left(\varphi^q\right)_i - k^2(n-1)i\sum_{i=1}^{n}\int v^r\left(\varphi^q\right)_i v_{\bar{i}} v_0 \\
&\quad - k^2(r+2k+2)(n-1)i\int v^{r-1}\varphi^q|\nabla v|^2 v_0. \qquad (2.25)
\end{aligned}
$$

Secondly using (2.12), integration by parts and (2.11)

$$
\begin{aligned}
V &= -\sum_{i,j=1}^{n}\int E^u_{i\bar{j}}u_{\bar{i}}\left(\varphi^q\right)_j v^{r+2k+2} \\
&= -\sum_{i,j=1}^{n}\int \left(u_{i\bar{j}} - \frac{1}{n}\sum_{\gamma=1}^{n}u_{\gamma\bar{\gamma}}\delta_{i\bar{j}}\right)u_{\bar{i}}\left(\varphi^q\right)_j v^{r+2k+2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n}\int\left[\left(|\nabla u|^2\right)_{\bar{j}} - \sum_{i=1}^{n} u_i u_{i\bar{j}}\right]\left(\varphi^q\right)_j v^{r+2k+2} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\int\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}} u_{\bar{i}}\left(\varphi^q\right)_i v^{r+2k+2} \\
&= \sum_{j=1}^{n}\int|\nabla u|^2 v^{r+2k+2}\left(\varphi^q\right)_{j\bar{j}} + (r+2k+2)\sum_{j=1}^{n}\int|\nabla u|^2 v^{r+2k+1}\left(\varphi^q\right)_j v_{\bar{j}} \\
&\quad + \sum_{i,j=1}^{n}\int u_i u_{i\bar{j}}\left(\varphi^q\right)_j v^{r+2k+2} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\int\left[-kniv^{-k-1}v_0 - \frac{h}{2}v^{-pk}\right]\left[-kv^{-k-1}v_{\bar{i}}\right]\left(\varphi^q\right)_i v^{r+2k+2}.
\end{aligned}
\tag{2.26}
$$

In the above equality, we need to express the term $\sum_{i,j=1}^{n}\int u_i u_{i\bar{j}}(\varphi^q)_j v^{r+2k+2} =: V_3$ in two kinds of the form V_3^1 and V_3^2. The reason will be pointed out at the end of Section 3. On the one hand, from integration by parts for subindex j,

$$
\begin{aligned}
V_3 = V_3^1 &= \sum_{i,j=1}^{n}\int u_i u_{i\bar{j}}\left(\varphi^q\right)_j v^{r+2k+2} \\
&= -\sum_{i,j=1}^{n}\int u_{ij}u_{\bar{i}\bar{j}}\varphi^q v^{r+2k+2} - \sum_{i,j=1}^{n}\int u_i u_{\bar{i}\bar{j}j}\varphi^q v^{r+2k+2} \\
&\quad - \sum_{i,j=1}^{n}\int u_i u_{\bar{i}\bar{j}}\varphi^q\left(v^{r+2k+2}\right)_j \\
&= -\sum_{i,j=1}^{n}\int\left[-kv^{-k-1}v_{ij} + k(k+1)v^{-k-2}v_i v_j\right] \\
&\quad \times\left[-kv^{-k-1}v_{\bar{i}\bar{j}} + k(k+1)v^{-k-2}v_{\bar{i}}v_{\bar{j}}\right]\varphi^q v^{r+2k+2} \\
&\quad - \sum_{i,j=1}^{n}\int u_i\left[u_{\bar{j}j\bar{i}} - 2i\delta_{\bar{i}j}u_{\bar{j}0}\right]\varphi^q v^{r+2k+2} \\
&\quad + (r+2k+2)k\sum_{i,j=1}^{n}\int\left[-kv^{-k-1}v_{\bar{i}\bar{j}} + k(k+1)v^{-k-2}v_{\bar{i}}v_{\bar{j}}\right]\varphi^q v^{r+k}v_i v_j,
\end{aligned}
$$

then

$$
\begin{aligned}
V_3 &= V_3^1 \\
&= -k^2\sum_{i,j=1}^{n}\int v^r\varphi^q|v_{ij}|^2 + k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q\left(v_{ij}v_{\bar{i}}v_{\bar{j}} + v_{\bar{i}\bar{j}}v_i v_j\right) \\
&\quad + \left[k^2(k+1)(r+2k+2) - k^2(k+1)^2\right]\int v^{r-2}\varphi^q|\nabla v|^4
\end{aligned}
$$

$$
\begin{aligned}
&- k^2(r+2k+2)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q v_{i\bar{j}} v_i v_j \\
&- \sum_{i=1}^{n}\int u_i\left[\left(-niu_0 - \frac{h}{2}u^p\right)_{\bar{i}} - 2iu_{\bar{i}0}\right]\varphi^q v^{r+2k+2},
\end{aligned} \tag{2.27}
$$

where the last term, according to (2.23) and (2.24), can be expressed as

$$
\begin{aligned}
&- \sum_{i=1}^{n}\int u_i\left[\left(-niu_0 - \frac{h}{2}u^p\right)_{\bar{i}} - 2iu_{\bar{i}0}\right]\varphi^q v^{r+2k+2} \\
&= (n+2)i\sum_{i=1}^{n}\int u_i u_{0\bar{i}}\varphi^q v^{r+2k+2} + \frac{1}{2}\sum_{i=1}^{n}\int u_i\left(hu^p\right)_{\bar{i}}\varphi^q v^{r+2k+2} \\
&= n(n+2)k^2\int v^r\varphi^q v_0^2 - \left[\frac{k(n+2)}{2} - \frac{nk}{2}\right]i\int hv^{r+k+1-pk}\varphi^q v_0 \\
&\quad + \frac{k(r+2k+2)}{2}\int hv^{r+k-pk}\varphi^q|\nabla v|^2 + \frac{1}{4}\int h^2 v^{r+2k+2-2pk}\varphi^q \\
&\quad + \frac{k}{2}\sum_{i=1}^{n}\int hv^{r+k+1-pk}v_i\left(\varphi^q\right)_{\bar{i}} - k^2(n+2)i\sum_{i=1}^{n}\int v^r\left(\varphi^q\right)_{\bar{i}} v_i v_0 \\
&\quad - k^2(r+2k+2)(n+2)i\int v^{r-1}\varphi^q|\nabla v|^2 v_0.
\end{aligned} \tag{2.28}
$$

Putting (2.28) into (2.27), we get

$$
\begin{aligned}
V_3 = V_3^1 = {}& k^2(k+1)(r+k+1)\int v^{r-2}\varphi^q|\nabla v|^4 - k^2\sum_{i,j=1}^{n}\int v^r\varphi^q|v_{ij}|^2 \\
&+ k^2(k+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q v_{ij} v_{\bar{i}} v_{\bar{j}} \\
&- k^2(r+k+1)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q v_{i\bar{j}} v_i v_j \\
&+ n(n+2)k^2\int v^r\varphi^q v_0^2 + \frac{1}{4}\int h^2 v^{r+2k+2-2pk}\varphi^q \\
&+ \frac{k(r+2k+2)}{2}\int hv^{r+k-pk}\varphi^q|\nabla v|^2 \\
&+ \frac{k}{2}\sum_{i=1}^{n}\int hv^{r+2k+1-pk}v_i\left(\varphi^q\right)_{\bar{i}} - k^2(n+2)i\sum_{i=1}^{n}\int v^r\left(\varphi^q\right)_{\bar{i}} v_i v_0 \\
&- ki\int hv^{r+k+1-pk}\varphi^q v_0 \\
&- (r+2k+2)k^2(n+2)i\int v^{r-1}\varphi^q|\nabla v|^2 v_0.
\end{aligned} \tag{2.29}
$$

On the other hand, using integration by parts for subindex $\bar{i}$,

$$\begin{aligned}
V_3 = V_3^2 &= \sum_{i,j=1}^{n} \int u_i u_{\bar{j}\bar{i}} (\varphi^q)_j v^{r+2k+2} \\
&= -\sum_{i,j=1}^{n} \int u_{i\bar{i}} u_{\bar{j}} (\varphi^q)_j v^{r+2k+2} - \sum_{i,j=1}^{n} \int u_i u_{\bar{j}} (\varphi^q)_{j\bar{i}} v^{r+2k+2} \\
&\quad - \sum_{i,j=1}^{n} \int u_i u_{\bar{j}} (\varphi^q)_j (v^{r+2k+2})_{\bar{i}} \\
&= -\sum_{j=1}^{n} \int \left[-kniv^{-k-1} v_0 - \frac{h}{2} v^{-pk} \right] (-kv^{-k-1} v_{\bar{j}}) (\varphi^q)_j v^{r+2k+2} \\
&\quad - \sum_{i,j=1}^{n} \int k^2 v^{-2k-2} v_i v_{\bar{j}} (\varphi^q)_{j\bar{i}} v^{r+2k+2} \\
&\quad - \sum_{i,j=1}^{n} \int k^2 (r+2k+2) v^{-2k-2} v_i v_{\bar{j}} (\varphi^q)_j v^{r+2k+1} v_{\bar{i}} \\
&= -k^2 ni \sum_{i=1}^{n} v^r (\varphi^q)_i v_{\bar{i}} v_0 - \frac{k}{2} \sum_{i=1}^{n} \int h v^{r+k+1-pk} (\varphi^q)_i v_{\bar{i}} \\
&\quad - k^2 \sum_{i,j=1}^{n} \int v^r (\varphi^q)_{j\bar{i}} v_i v_{\bar{j}} \\
&\quad - k^2 (r+2k+2) \sum_{i=1}^{n} \int v^{r-1} |\nabla v|^2 v_{\bar{i}} (\varphi^q)_i .
\end{aligned} \tag{2.30}$$

Thus combining (2.29) and (2.30) with (2.26),

$$\begin{aligned}
V &= k^2 \sum_{i=1}^{n} \int v^r |\nabla v|^2 (\varphi^q)_{i\bar{i}} + \frac{k}{2n} \sum_{i=1}^{n} \int h v^{r+k+1-pk} (\varphi^q)_i v_{\bar{i}} \\
&\quad + k^2 (r+2k+2) \sum_{i=1}^{n} \int v^{r-1} |\nabla v|^2 (\varphi^q)_i v_{\bar{i}} \\
&\quad + k^2 i \sum_{i=1}^{n} \int v^r (\varphi^q)_i v_{\bar{i}} v_0 + \eta V_3^1 + (1-\eta) V_3^2,
\end{aligned} \tag{2.31}$$

where η is to be determined.

Finally in Step II, we calculate VI as follows

$$\begin{aligned}
VI &= -\sum_{i,j=1}^{n} \int E^u_{i\bar{j}} u_{\bar{i}} \varphi^q (v^{r+2k+2})_j \\
&= -\sum_{i,j=1}^{n} \int \left[u_{i\bar{j}} - \frac{1}{n} \sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}} \delta_{i\bar{j}} \right] u_{\bar{i}} \varphi^q (v^{r+2k+2})_j
\end{aligned}$$

$$= -(r+2k+2)\sum_{j=1}^{n}\int\left[\left(|\nabla u|^2\right)_{\bar{j}} - \sum_{i=1}^{n} u_i u_{\bar{i}\bar{j}}\right]\varphi^q v^{r+2k+1} v_j$$

$$+ \frac{r+2k+2}{n}\sum_{i=1}^{n}\int\sum_{\gamma=1}^{n} u_{\gamma\bar{\gamma}} u_{\bar{i}} \varphi^q v^{r+2k+1} v_i. \tag{2.32}$$

Similar to the process in (2.26), we obtain

$$VI = k^2(r+2k+2)(r+2k+1)\int v^{r-2}\varphi^q|\nabla v|^4$$

$$+ \frac{k(r+2k+2)}{2}\left(1+\frac{1}{n}\right)\int h v^{r+k-pk}\varphi^q|\nabla v|^2$$

$$+ k^2(r+2k+2)(n+1)i\int v^{r-1}\varphi^q|\nabla v|^2 v_0$$

$$+ k^2(r+2k+2)\sum_{i=1}^{n}\int v^{r-1}\left(\varphi^q\right)_{\bar{i}}|\nabla v|^2 v_i$$

$$+ k^2(r+2k+2)\sum_{i,j=1}^{n}\int v^{r-1}\varphi^q v_{\bar{i}\bar{j}} v_i v_j. \tag{2.33}$$

3. Proof of Theorem 1

Noting that the integral $\sum_{i,j=1}^{n}\int|E^u_{i\bar{j}}|^2\varphi^q v^{r+2k+2}$ is real, so

$$IV + V + VI = \overline{IV} + \overline{V} + \overline{VI}. \tag{3.1}$$

Combining Steps I and II in Section 2, we should have

$$I + II + III = \frac{IV + V + VI + \overline{IV} + \overline{V} + \overline{VI}}{2}. \tag{3.2}$$

From the above we finally obtain

$$k^2\sum_{i,j=1}^{n}\int\left|E^v_{i\bar{j}}\right|^2\varphi^q v^r + \lambda_1\int v^{r-2}\varphi^q|\nabla v|^4 + \lambda_2\int h^2 v^{r+2k+2-2pk}\varphi^q$$

$$+ \lambda_3\int v^r\varphi^q v_0^2 + \lambda_4\sum_{i,j=1}^{n}\int v^r\varphi^q|v_{ij}|^2 + \lambda_5\int h v^{r+k-pk}\varphi^q|\nabla v|^2$$

$$= \lambda_6\sum_{j=1}^{n}\int v^{r-1}|\nabla v|^2\left[\left(\varphi^q\right)_j v_{\bar{j}} + \left(\varphi^q\right)_{\bar{j}} v_j\right] + \lambda_7\int v^r|\nabla v|^2\Delta_H\left(\varphi^q\right)$$

$$+ \lambda_8\sum_{i,j=1}^{n}\int v^r\left[\left(\varphi^q\right)_{ji} v_i v_{\bar{j}} + \left(\varphi^q\right)_{\bar{j}\bar{i}} v_{\bar{i}} v_j\right] + \lambda_9\sum_{j=1}^{n}\int v^r\left[\left(\varphi^q\right)_j v_{\bar{j}} - \left(\varphi^q\right)_{\bar{j}} v_j\right] v_0$$

$$+\lambda_{10}\sum_{i,j=1}^{n}\int v^{r-1}\varphi^{q}(v_{\bar{i}}v_{\bar{j}}v_{ij}+v_{i}v_{j}v_{\bar{i}\bar{j}})$$

$$+\lambda_{11}\sum_{j=1}^{n}\int hv^{r+k+1-pk}\big[(\varphi^{q})_{j}v_{\bar{j}}+(\varphi^{q})_{\bar{j}}v_{j}\big], \tag{3.3}$$

where

$$\begin{aligned}\lambda_1 &= k^2(k+1)\left[(k+1)\left(3+\frac{1}{n}\right)+2(r-1)\right]-k^2(k+1)(r+k+1)\eta\\ &\quad -k^2(r+2k+2)(r+2k+1)\\ &= k^2\left[\left(\frac{1}{n}-1-\eta\right)(k+1)^2-(2+\eta)(k+1)r-r(r-1)\right],\end{aligned} \tag{3.4}$$

$$\lambda_2 = -\frac{n-1}{4n}-\frac{\eta}{4}, \tag{3.5}$$

$$\lambda_3 = n(n-1)k^2-n(n+2)k^2\eta = nk^2\big[(n-1)-(n+2)\eta\big], \tag{3.6}$$

$$\lambda_4 = k^2\eta, \tag{3.7}$$

$$\begin{aligned}\lambda_5 &= \left(1+\frac{1}{n}\right)k(k+1)-\frac{k(r+2k+2)(n-1)}{2n}\\ &\quad -\frac{k(r+2k+2)}{2}\eta-\left(1+\frac{1}{n}\right)\frac{k(r+2k+2)}{2}\\ &= \left(1+\frac{1}{n}\right)k(k+1)-k(r+2k+2)\left(1+\frac{\eta}{2}\right),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\lambda_6 &= -k^2(k+1)+\frac{k^2(r+2k+2)}{2}\\ &\quad -\frac{k^2(r+2k+2)}{2}(1-\eta)+\frac{k^2(r+2k+2)}{2}\\ &= k^2\left[\frac{(r+2k+2)}{2}(1+\eta)-(k+1)\right],\end{aligned} \tag{3.9}$$

$$\lambda_7 = \frac{k^2}{2}, \tag{3.10}$$

$$\lambda_8 = \frac{-k^2(1-\eta)}{2}, \tag{3.11}$$

$$\begin{aligned}\lambda_9 &= \left[-\frac{k^2(n-1)}{2}+\frac{k^2}{2}+\frac{k^2(n+2)}{2}\eta-\frac{k^2n}{2}(1-\eta)\right]i\\ &= k^2\big[(n+1)\eta-(n-1)\big]i,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\lambda_{10} &= -k^2(k+1)+\frac{k^2(k+1)-k^2(r+k+1)}{2}\eta+\frac{k^2(r+2k+2)}{2}\\ &= \frac{k^2r}{2}(1-\eta),\end{aligned} \tag{3.13}$$

$$\lambda_{11} = \frac{k(n-1)}{4n} + \frac{k}{4n} + \frac{k}{4}\eta - \frac{k}{4}(1-\eta) = \frac{k}{2}\eta. \tag{3.14}$$

At first we deal with $\lambda_5 \int h v^{r+k-pk} \varphi^q |\nabla v|^2$. Since

$$\begin{aligned}
&\sum_{i=1}^n \int \left(hu^p\right)_i u_{\bar{i}} \varphi^q v^{r+2k+2} \\
&\quad = \sum_{i=1}^n \int h_i u_{\bar{i}} u^p \varphi^q v^{r+2k+2} + \sum_{i=1}^n \int h\left(u^p\right)_i u_{\bar{i}} \varphi^q v^{r+2k+2} \\
&\quad = pk^2 \int h v^{r+k-pk} \varphi^q |\nabla v|^2 - k \sum_{i=1}^n \int v^{r+k+1-pk} \varphi^q v_{\bar{i}} h_i,
\end{aligned} \tag{3.15}$$

and recalling (2.24), we have

$$\begin{aligned}
&\int h v^{r+k-pk} \varphi^q |\nabla v|^2 \\
&\quad = -\frac{1}{2k(r+2k+2-pk)} \int h^2 v^{r+2k+2-2pk} \varphi^q \\
&\qquad - \frac{1}{2(r+2k+2-pk)} \sum_{i=1}^n \int v^{r+k+1-pk} \varphi^q [h_i v_{\bar{i}} + h_{\bar{i}} v_i] \\
&\qquad - \frac{1}{2(r+2k+2-pk)} \sum_{i=1}^n \int v^{r+k+1-pk} h\left[v_{\bar{i}}\left(\varphi^q\right)_i + v_i\left(\varphi^q\right)_{\bar{i}}\right].
\end{aligned} \tag{3.16}$$

Here using the fact that the integral is real again. Putting (3.16) into $\lambda_5 \int h v^{r+k-pk} \varphi^q |\nabla v|^2$, the equality (3.3) is changed into

$$\begin{aligned}
&k^2 \sum_{i,j=1}^n \int \left|E^v_{i\bar{j}}\right|^2 \varphi^q v^r + \lambda_1 \int v^{r-2} \varphi^q |\nabla v|^4 + \lambda_2' \int h^2 v^{r+2k+2-2pk} \varphi^q \\
&\qquad + \lambda_3 \int v^r \varphi^q v_0^2 + \lambda_4 \sum_{i,j=1}^n \int v^r \varphi^q |v_{ij}|^2 \\
&\quad = \lambda_6 \sum_{j=1}^n \int v^{r-1} |\nabla v|^2 \left[\left(\varphi^q\right)_j v_{\bar{j}} + \left(\varphi^q\right)_{\bar{j}} v_j\right] + \lambda_7 \int v^r |\nabla v|^2 \Delta_H\left(\varphi^q\right) \\
&\qquad + \lambda_8 \sum_{i,j=1}^n \int v^r \left[\left(\varphi^q\right)_{j\bar{i}} v_i v_{\bar{j}} + \left(\varphi^q\right)_{\bar{j}i} v_{\bar{i}} v_j\right] + \lambda_9 \sum_{j=1}^n \int v^r \left[\left(\varphi^q\right)_j v_{\bar{j}} - \left(\varphi^q\right)_{\bar{j}} v_j\right] v_0 \\
&\qquad + \lambda_{10} \sum_{i,j=1}^n \int v^{r-1} \varphi^q (v_{\bar{i}} v_{\bar{j}} v_{ij} + v_i v_j v_{\bar{i}\bar{j}})
\end{aligned}$$

$$+\lambda_{11}'\sum_{j=1}^{n}\int hv^{r+k+1-pk}\big[(\varphi^q)_j v_{\bar j}+(\varphi^q)_{\bar j}v_j\big]$$

$$+\lambda_{11}''\sum_{j=1}^{n}\int \varphi^q v^{r+k+1-pk}[h_j v_{\bar j}+h_{\bar j}v_j], \tag{3.17}$$

where

$$\begin{aligned}\lambda_2' &= -\frac{1}{2k(r+2k+2-pk)}\lambda_5+\lambda_2\\ &=: \lambda_5'+\lambda_2\\ &= \frac{(1+\frac{1}{n})r+(1+\eta-\frac{1}{n})pk}{4(r+2k+2-pk)},\end{aligned} \tag{3.18}$$

$$\lambda_{11}'=\lambda_{11}-k\lambda_5', \qquad \lambda_{11}''=-k\lambda_5', \tag{3.19}$$

and the others as before.

Proof of Theorem 1. Now we choose

$$k\neq 0, \qquad p>1, \qquad h\geqslant 0. \tag{3.20}$$

We will use the identity (3.17) to conclude $v\equiv 0$, thus $u\equiv 0$. Our goal is to make the left-hand side of (3.17) to be positive, which need $\lambda_1, \lambda_2', \lambda_3$ and λ_4 all be nonnegative, and at the same time, the exponent of v in positive integrals to be nonnegative at least, which needs $r\geqslant 0$ at least since the first term is positive for $k\neq 0$. On the other hand, we need that the right-hand side of (3.17) can be controlled by the terms in the left-hand side and the cut-off function φ with its derivatives, which can be realized as follows by repeatedly using Young's inequality

$$ab\leqslant \epsilon^t\frac{a^t}{t}+\frac{1}{\epsilon^s}\frac{b^s}{s}, \qquad \frac{1}{t}+\frac{1}{s}=1, \tag{3.21}$$

valid for positive numbers a, b, t, s, and ϵ. In the rest of the paper ε will denote a sufficiently small positive number, and C a positive constant which may vary from line to line. Then

$$\begin{aligned}&\lambda_6\sum_{j=1}^{n}\int v^{r-1}|\nabla v|^2\big[(\varphi^q)_j v_{\bar j}+(\varphi^q)_{\bar j}v_j\big]\\ &= q\lambda_6\sum_{j=1}^{n}\int v^{r-1}|\nabla v|^2\varphi^{q-1}(\varphi_j v_{\bar j}+\varphi_{\bar j}v_j)\\ &\leqslant \varepsilon\int v^{r-2}\varphi^q|\nabla v|^4+C\int v^{r+2}\varphi^{q-4}|\nabla\varphi|^4.\end{aligned} \tag{3.22}$$

Since $\varphi\leqslant 1$, $\varphi^{q-1}\leqslant\varphi^{q-2}$,

$$\begin{aligned}
&\lambda_7 \int v^r |\nabla v|^2 \Delta_H(\varphi^q) \\
&\quad = \lambda_7 \int v^r |\nabla v|^2 \bigl[q\varphi^{q-1}\Delta_H\varphi + 2q(q-1)\varphi^{q-2}|\nabla\varphi|^2\bigr] \\
&\quad \leqslant \varepsilon \int v^{r-2}\varphi^q |\nabla v|^4 + C \int v^{r+2}\varphi^{q-4}\bigl(|\Delta_H\varphi| + |\nabla\varphi|^2\bigr)^2.
\end{aligned} \tag{3.23}$$

Similarly

$$\begin{aligned}
&\lambda_8 \sum_{i,j=1}^{n} \int v^r \bigl[(\varphi^q)_{j\bar{i}} v_i v_{\bar{j}} + (\varphi^q)_{\bar{j}i} v_{\bar{i}} v_j\bigr] \\
&\quad \leqslant \varepsilon \int v^{r-2}\varphi^q |\nabla v|^4 + C \int v^{r+2}\varphi^{q-4}\bigl(|\nabla^2\varphi| + |\nabla\varphi|^2\bigr)^2.
\end{aligned} \tag{3.24}$$

Now we will explain why we can't do case $n = 1$ pointed in Remark 1. Since we need

$$\lambda_3 = nk^2\bigl[(n-1) - (n+2)\eta\bigr] \geqslant 0, \tag{3.25}$$

and

$$\lambda_4 = k^2\eta \geqslant 0, \tag{3.26}$$

it gives

$$0 \leqslant \eta \leqslant (n-1)/(n+2). \tag{3.27}$$

If $n = 1$, it leads to $\eta = 0$, then we know from (3.12) that $\lambda_9 = 0$. At the same time $\lambda_{10} \neq 0$ except for $r = 0$. However the case $r = 0$ can be excluded. In fact, because λ_7 and λ_8 are not equal to zero, we need $\lambda_1 > 0$ to control the terms $\varepsilon \int v^{r-2}\varphi^q |\nabla v|^4$ in (3.23) and (3.24). Recalling (3.4), $n = 1$, $\eta = 0$, $r = 0$ gives $\lambda_1 = 0$, a contradiction. Now since $\lambda_{10} \neq 0$, we have

$$\begin{aligned}
&\lambda_{10} \sum_{i,j=1}^{n} \int v^{r-1}\varphi^q (v_{\bar{i}} v_{\bar{j}} v_{ij} + v_i v_j v_{\bar{i}\bar{j}}) \\
&\quad \leqslant a' \sum_{i,j=1}^{n} \int v^r \varphi^q |v_{ij}|^2 + b' \int v^{r-2}\varphi^q |\nabla v|^4,
\end{aligned} \tag{3.28}$$

valid for $a' > 0, b' > 0$ and

$$a'b' \geqslant \lambda_{10}^2 = \frac{k^4 r^2 (1-\eta)^2}{4}. \tag{3.29}$$

Here it appears a positive term $a' \sum_{i,j=1}^{n} \int v^r \varphi^q |v_{ij}|^2$. To control it we need $\lambda_4 \geqslant a'$, but it's impossible for $\eta = 0$. So we can't realize our goal at the beginning of our proof, it means that we can't deal with $n = 1$ using our method at present.

In the following, we consider $n \geqslant 2$. Again by (3.12) we know that $\lambda_9 \neq 0$. Then

$$\begin{aligned}
&\lambda_9 \sum_{j=1}^n \int v^r\big[\big(\varphi^q\big)_j v_{\bar{j}} - \big(\varphi^q\big)_{\bar{j}} v_j\big] v_0 \\
&\quad\leqslant \varepsilon \int v^r v_0^2 \varphi^q + C \int v^r \varphi^{q-2} |\nabla v|^2 |\nabla \varphi|^2 \\
&\quad\leqslant \varepsilon \int v^r v_0^2 \varphi^q + \varepsilon \int v^{r-2} \varphi^q |\nabla v|^4 + C \int v^{r+2} \varphi^{q-4} |\nabla \varphi|^4.
\end{aligned} \tag{3.30}$$

It demands $\lambda_3 > 0$ in return to control $\varepsilon \int v^r v_0^2 \varphi^q$. It results

$$0 \leqslant \eta < (n-1)/(n+2). \tag{3.31}$$

To ensure $\lambda_1 > 0$ as before, r cannot be zero, and λ_{10} also. The same way as above we obtain (3.28) and (3.29).

Using integration by parts,

$$\begin{aligned}
&\lambda'_{11} \sum_{j=1}^n \int h v^{r+k+1-pk}\big[\big(\varphi^q\big)_j v_{\bar{j}} + \big(\varphi^q\big)_{\bar{j}} v_j\big] \\
&= \frac{\lambda'_{11}}{r+k+2-pk} \sum_{j=1}^n \int h\big[\big(\varphi^q\big)_j \big(v^{r+k+2-pk}\big)_{\bar{j}} + \big(\varphi^q\big)_{\bar{j}} \big(v^{r+k+2-pk}\big)_j\big] \\
&= -\frac{\lambda'_{11}}{r+k+2-pk} \int v^{r+k+2-pk}\left[\sum_{j=1}^n \big[\big(\varphi^q\big)_j h_{\bar{j}} + \big(\varphi^q\big)_{\bar{j}} h_j\big] + h\Delta_H\big(\varphi^q\big)\right] \\
&= -\frac{\lambda'_{11} q}{r+k+2-pk} \sum_{j=1}^n \int v^{r+k+2-pk} h \varphi^{q-1}\big[\varphi_j (\log h)_{\bar{j}} + \varphi_{\bar{j}} (\log h)_j\big] \\
&\quad - \frac{\lambda'_{11}}{r+k+2-pk} \int v^{r+k+2-pk} h\big[q\varphi^{q-1}\Delta_H \varphi \\
&\quad + 2q(q-1)\varphi^{q-2}|\nabla \varphi|^2\big].
\end{aligned} \tag{3.32}$$

And

$$\begin{aligned}
&\lambda''_{11} \sum_{j=1}^n \int \varphi^q v^{r+k+1-pk}[h_j v_{\bar{j}} + h_{\bar{j}} v_j] \\
&\quad= -\frac{\lambda''_{11} q}{r+k+2-pk} \sum_{j=1}^n \int v^{r+k+2-pk} h \varphi^{q-1}\big[\varphi_j (\log h)_{\bar{j}} + \varphi_{\bar{j}} (\log h)_j\big] \\
&\qquad - \frac{\lambda''_{11}}{r+k+2-pk} \int v^{r+k+2-pk} \varphi^q \Delta_H h \\
&\quad\leqslant -\frac{\lambda''_{11} q}{r+k+2-pk} \sum_{j=1}^n \int v^{r+k+2-pk} h \varphi^{q-1}\big[\varphi_j (\log h)_{\bar{j}} + \varphi_{\bar{j}} (\log h)_j\big],
\end{aligned} \tag{3.33}$$

if

$$\frac{\lambda''_{11}}{r+k+2-pk}\Delta_H h \geqslant 0. \tag{3.34}$$

Since we need $\eta \geqslant 0$ such that $\lambda_4 = k^2\eta \geqslant 0$, it makes at the same time $\lambda_2 < 0$ from (3.5) and $\lambda'_5 \geqslant -\lambda_2 > 0$ to ensure $\lambda'_2 \geqslant 0$ from (3.18). Then assumption (3.34), with $\lambda''_{11} = -k\lambda'_5$, is equal to

$$\frac{k}{r+k+2-pk}\Delta_H h \leqslant 0. \tag{3.35}$$

Thus combining (3.32) and (3.33) we find

$$\begin{aligned}
&\lambda'_{11}\sum_{j=1}^{n}\int h v^{r+k+1-pk}\big[\big(\varphi^q\big)_j v_{\bar j} + \big(\varphi^q\big)_{\bar j} v_j\big] \\
&\quad + \lambda''_{11}\sum_{j=1}^{n}\int \varphi^q v^{r+k+1-pk}[h_j v_{\bar j} + h_{\bar j} v_j] \\
&\leqslant C\int v^{r+k+2-pk} h\varphi^{q-2}\big[|\nabla \log h||\nabla\varphi| + |\Delta_H\varphi| + |\nabla\varphi|^2\big]
\end{aligned} \tag{3.36}$$

under the assumption (3.35).

Inserting (3.22)–(3.36) into (3.17), we easily get

$$\begin{aligned}
&k^2\sum_{i,j=1}^{n}\int \big|E^v_{i\bar j}\big|^2\varphi^q v^r + (\lambda_1 - b' - \varepsilon)\int v^{r-2}\varphi^q|\nabla v|^4 \\
&\quad + \lambda'_2\int h^2 v^{r+2k+2-2pk}\varphi^q + (\lambda_3 - \varepsilon)\int v^r\varphi^q v_0^2 \\
&\quad + (\lambda_4 - a')\sum_{i,j=1}^{n}\int v^r\varphi^q|v_{ij}|^2 \\
&\leqslant C\int v^{r+k+2-pk} h\varphi^{q-2}\big[|\nabla \log h||\nabla\varphi| + |\Delta_H\varphi| + |\nabla\varphi|^2\big] \\
&\quad + C\int v^{r+2}\varphi^{q-4}\big[\big(|\Delta_H\varphi| + |\nabla\varphi|^2\big)^2 + |\nabla\varphi|^4\big].
\end{aligned} \tag{3.37}$$

Now we deal with the right-hand side of (3.37) using Young's inequality again. Choosing $t_1 = \frac{r+2k+2-2pk}{r+k+2-pk} > 1$, $s_1 = \frac{r+2k+2-2pk}{k-pk} > 1$ in (3.21), we have

$$\begin{aligned}
&\int v^{r+k+2-pk} h\varphi^{q-2}\big[|\nabla \log h||\nabla\varphi| + |\Delta_H\varphi| + |\nabla\varphi|^2\big] \\
&\leqslant \varepsilon\int h^2 v^{r+2k+2-2pk}\varphi^q \\
&\quad + C\int h^{-\frac{r+2}{k-pk}}\varphi^{\mu_1}\big[|\nabla \log h||\nabla\varphi| + |\Delta_H\varphi| + |\nabla\varphi|^2\big]^{s_1}
\end{aligned} \tag{3.38}$$

with $\mu_1 = (q-2-\frac{q}{t_1})s_1$.

Choosing $t_2 = \frac{r+2k+2-2pk}{r+2} > 1$, $s_2 = \frac{1}{2}s_1 = \frac{r+2k+2-2pk}{2(k-pk)} > 1$ in (3.21), we have

$$\begin{aligned}
&\int v^{r+2}\varphi^{q-4}\big[\big(|\Delta_H\varphi| + |\nabla\varphi|^2\big)^2 + |\nabla\varphi|^4\big] \\
&\quad \leqslant \varepsilon \int h^2 v^{r+2k+2-2pk}\varphi^q \\
&\qquad + C\int h^{-\frac{r+2}{k-pk}}\varphi^{\mu_2}\big[\big(|\Delta_H\varphi| + |\nabla\varphi|^2\big)^2 + |\nabla\varphi|^4\big]^{s_2}
\end{aligned} \tag{3.39}$$

with $\mu_2 = (q - 4 - \frac{q}{t_2})s_2 = \mu_1$.

Thus

$$\begin{aligned}
&k^2\sum_{i,j=1}^{n}\int \big|E^{v}_{i\bar{j}}\big|^2\varphi^q v^r + (\lambda_1 - b' - \varepsilon)\int v^{r-2}\varphi^q|\nabla v|^4 \\
&\quad + \big(\lambda_2' - \varepsilon\big)\int h^2 v^{r+2k+2-2pk}\varphi^q + (\lambda_3 - \varepsilon)\int v^r\varphi^q v_0^2 \\
&\quad + (\lambda_4 - a')\sum_{i,j=1}^{n}\int v^r\varphi^q|v_{ij}|^2 \\
&\leqslant C\int h^{-\frac{r+2}{k-pk}}\varphi^{\mu_1}\big[|\nabla\log h||\nabla\varphi| + |\Delta_H\varphi| + |\nabla\varphi|^2\big]^{s_1} \\
&\quad + C\int h^{-\frac{r+2}{k-pk}}\varphi^{\mu_2}\big[\big(|\Delta_H\varphi| + |\nabla\varphi|^2\big)^2 + |\nabla\varphi|^4\big]^{s_2}.
\end{aligned} \tag{3.40}$$

We pick $\Omega = \{x \in H^n \subset \mathbb{R}^{2n+1}\colon\ |x| \leqslant R\}$ and

$$\varphi(x) = 1 \quad \text{for } |x| \leqslant \frac{1}{2}R,$$

$$0 \leqslant \varphi(x) \leqslant 1 \quad \text{for } \frac{1}{2}R < |x| \leqslant R, \tag{3.41}$$

$$|\nabla\varphi| \leqslant \frac{C}{R}, \qquad \big|\nabla^2\varphi\big| \leqslant \frac{C}{R^2} \quad \text{for } \frac{1}{2}R \leqslant |x| \leqslant R. \tag{3.42}$$

Moreover we choose q large enough such that $\mu_1 = \mu_2 \geqslant 0$ to make $\varphi^{\mu_1} = \varphi^{\mu_2} \leqslant 1$. Together with conditions (1.7) and (1.8), and noting that the homogeneous dimension is $2n+2$, we have

$$\text{the right-hand side of (3.40)} \leqslant CR^{-2s_1 - \frac{r+2}{k-pk}\sigma + 2n+2} \tag{3.43}$$

under the assumption

$$\frac{r+2}{k(1-p)} > 0. \tag{3.44}$$

In fact from above $r > 0$, $p > 1$, it implies that $k < 0$. In return, it guarantees that t_1, s_1, t_2, s_2 are all more than 1. And assumption (3.35) also holds with the condition (1.6). It yields form (3.40) and (3.43)

$$\begin{aligned}
k^2 \sum_{i,j=1}^{n} \int \left|E_{i\bar{j}}^{v}\right|^2 \varphi^q v^r &+ (\lambda_1 - b' - \varepsilon) \int v^{r-2} \varphi^q |\nabla v|^4 \\
&+ \left(\lambda_2' - \varepsilon\right) \int h^2 v^{r+2k+2-2pk} \varphi^q + (\lambda_3 - \varepsilon) \int v^r \varphi^q v_0^2 \\
&+ (\lambda_4 - a') \sum_{i,j=1}^{n} \int v^r \varphi^q |v_{ij}|^2 \\
&\leqslant C R^{-2s_1 - \frac{r+2}{k-pk}\sigma + 2n + 2}. \qquad (3.45)
\end{aligned}$$

We can see that (3.45) implies $v \equiv 0$ and thus $u \equiv 0$ as $R \to +\infty$, on the conditions that

$$\lambda_1 - b' > 0, \qquad \lambda_2' > 0, \qquad \lambda_3 > 0, \qquad \lambda_4 - a' \geqslant 0, \qquad (3.46)$$

$$-2s_1 - \frac{r+2}{k-pk}\sigma + 2n + 2 < 0. \qquad (3.47)$$

To complete the proof of the theorem, we only need to pick suitable $k < 0$, $r > 0$, $p > 1$, η and then σ such that (3.46) and (3.47) hold. Firstly it needs $\lambda_4 \geqslant a'$, recalling (3.7) and (3.28), $a' > 0$ leads to $\eta > 0$. Secondly, according to (3.6), $\lambda_3 > 0$ needs $\eta < (n-1)/(n+2)$. In fact, λ_4 comes only from the term ηV_3^1. $0 < \eta < (n-1)/(n+2) < 1$ tells us why we should express the term V_3 in (2.31) as $\eta V_3^1 + (1-\eta) V_3^2$.

Again from (3.29), we can choose the best constant $a' = k^2\eta$ and $b' = k^4 r^2 (1-\eta)^2/(4a')$ such that

$$\begin{aligned}
\lambda_1 - b' &= k^2 \left[\left(\frac{1}{n} - 1 - \eta \right)(k+1)^2 - (2+\eta)(k+1)r - r(r-1) - \frac{r^2(1-\eta)^2}{4\eta} \right] \\
&> 0. \qquad (3.48)
\end{aligned}$$

And $\lambda_2' > 0$ implies that

$$\left[\left(1 + \frac{1}{n}\right) r + \left(1 + \eta - \frac{1}{n}\right) pk \right] (r + 2k + 2 - pk) > 0. \qquad (3.49)$$

Set $y = 1 + \frac{1}{k} < 1$, $\delta = -\frac{r}{k} > 0$. Then (3.48) is equivalent to the following inequality with quadratic polynomial about y at the left side:

$$\left(1 + \eta - \frac{1}{n}\right) y^2 - (1+\eta)\delta y + \left[\frac{(1+\eta)^2}{4\eta} \delta^2 - \delta \right] < 0. \qquad (3.50)$$

And (3.49) is equivalent to

$$\left[-\left(1 + \frac{1}{n}\right)\delta + \left(1 + \eta - \frac{1}{n}\right) p \right] (-\delta + 2y - p) > 0, \qquad (3.51)$$

i.e.

$$\delta > \frac{n + n\eta - 1}{n+1} p, \quad \text{if } 2y < \delta + p, \qquad (3.52)$$

$$\delta < \frac{n + n\eta - 1}{n+1} p, \quad \text{if } 2y > \delta + p. \qquad (3.53)$$

To realize (3.50) it is needed to choose suitable δ such that the discriminant of the quadratic polynomial more than zero, that is

$$\Delta_y := (1+\eta)^2\delta^2 - 4\left(1+\eta-\frac{1}{n}\right)\left[\frac{(1+\eta)^2}{4\eta}\delta^2 - \delta\right] > 0, \tag{3.54}$$

which gives

$$0 < \delta < \frac{4(n+n\eta-1)\eta}{(n-1)(1+\eta)^2}. \tag{3.55}$$

At the same time, y must be between the two roots of the quadratic polynomial, i.e.

$$y_1 := \frac{(1+\eta)\delta - \sqrt{\Delta_y}}{2(1+\eta-\frac{1}{n})} < y < \frac{(1+\eta)\delta + \sqrt{\Delta_y}}{2(1+\eta-\frac{1}{n})} := y_2. \tag{3.56}$$

Now considering (3.52), (3.53) and (3.56), we will compare y_1, y_2 and $\frac{\delta+p}{2}$. Let $X_1 = 2(\frac{\delta+p}{2})(1+\eta-\frac{1}{n}) - (1+\eta)\delta$, by condition (3.55),

$$\begin{aligned} X_1 &= p\left(1+\eta-\frac{1}{n}\right) - \frac{\delta}{n} \\ &> p\left(1+\eta-\frac{1}{n}\right)\left[1 - \frac{4\eta}{pn(1-\frac{1}{n})(1+\eta)^2}\right]. \end{aligned} \tag{3.57}$$

Set $w = \frac{\eta}{(1-\frac{1}{n})(1+\eta)^2}$, from calculation we know w is monotonically increasing w.r.t. η, and $\eta < (n-1)/(n+2)$, so $w < \frac{n(n+2)}{(2n+1)^2}$. $p > 1$, $n \geqslant 2$, thus

$$\begin{aligned} X_1 &> \left(1+\eta-\frac{1}{n}\right)\left[1 - \frac{4(n+2)}{(2n+1)^2}\right] \\ &= \frac{(1+\eta-\frac{1}{n})}{(2n+1)^2}\left[4n^2-7\right] \\ &> 0 > -\sqrt{\Delta_y}. \end{aligned} \tag{3.58}$$

So $\frac{\delta+p}{2} > y_1$ from (3.56). Next we set

$$\begin{aligned} X_2 &= X_1^2 - \Delta_y \\ &= \left[\frac{1}{n^2} + \left(1-\frac{1}{n}\right)\frac{(1+\eta)^2}{\eta}\right]\delta^2 - \left(1+\eta-\frac{1}{n}\right)\left(4+\frac{2p}{n}\right)\delta + p^2\left(1+\eta-\frac{1}{n}\right)^2. \end{aligned} \tag{3.59}$$

This is a quadratic polynomial of δ, its discriminant is

$$\Delta_\delta = 4\left(1+\eta-\frac{1}{n}\right)^2\left[\left(2+\frac{p}{n}\right)^2 - p^2\left[\frac{1}{n^2} + \left(1-\frac{1}{n}\right)\frac{(1+\eta)^2}{\eta}\right]\right]. \tag{3.60}$$

From calculations we know $\Delta_\delta < 0$ with conditions $p > 1$, $n \geqslant 2$ and $\eta < (n-1)/(n+2)$. It tells us that $X_2 > 0$, then $\frac{\delta+p}{2} > y_2$ again from (3.56). Till now we can choose y s.t. $y_1 < y < y_2 < \frac{\delta+p}{2}$, and any δ satisfying (3.55). At last we get the range of p by (3.52) and (3.55),

$$\begin{aligned} 1 < p < \frac{n+1}{n+n\eta-1}\delta &< \frac{4(n+1)\eta}{(n-1)(1+\eta)^2} \\ &= 4\frac{n+1}{n}w \\ &< \frac{4(n+1)(n+2)}{(2n+1)^2} \\ &= \frac{Q(Q+2)}{(Q-1)^2}. \end{aligned} \tag{3.61}$$

That is what we can get about p in Theorem 1. In fact, we can choose $\delta = 8(n-1)(n+1)/(2n+1)^2 - \theta$ for sufficiently small θ, and $y = 2n/(2n+1)$, then $k = -(2n+1)$, $r = 8(n-1)(n+1)/(2n+1) - (2n+1)\theta$. At last, we get σ satisfying (1.8) by (3.47). $\square$

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